ECS455 HW5 Q1 Markov Chain: Land of Oz

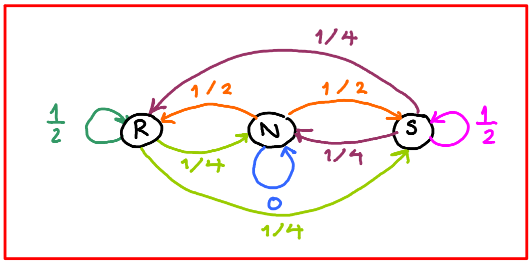
Tuesday, January 11, 2011 8:52 AM

(a)

As hinted, we draw three states.

Next, follow the description sentence by sentence to get the transition probabilities.

1. Never have two nice days in a row
2. If have a nice day, just as likely to have snow as rain the next day
3. If have snow or rain, they have an even chance of having the same the next day.
4. If there is change from snow or rain, only half of the time is this a change to a nice day.



Note that we set the color of each arrow and its label to match with the corresponding description above.

(b) The corresponding transition probability matrix is

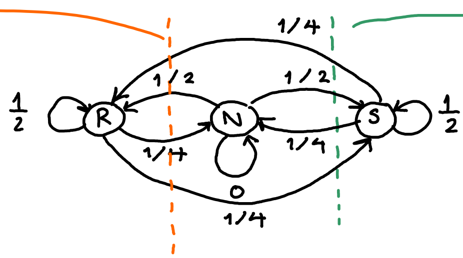
$$P = \begin{array}{c} \\ R \\ N \\ S \end{array} \begin{array}{c} \begin{array}{ccc} R & N & S \end{array} \\ \begin{bmatrix} 1/2 & 1/4 & 1/4 \\ 1/2 & 0 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{bmatrix} \end{array}$$

(c) (i) We first write two balance equations from the two boundaries shown below:

$$\frac{1}{2}p_N + \frac{1}{4}p_S = \frac{1}{4}p_R + \frac{1}{4}p_R$$

$$\frac{1}{2}p_N + \frac{1}{4}p_S = \frac{1}{2}p_R$$

$$-\frac{1}{2}p_R + \frac{1}{2}p_N + \frac{1}{4}p_S = 0$$



$$\frac{1}{4}p_S + \frac{1}{4}p_S = \frac{1}{2}p_N + \frac{1}{4}p_R$$

$$\frac{1}{2}p_S = \frac{1}{2}p_N + \frac{1}{4}p_R$$

$$\frac{1}{4}p_R + \frac{1}{2}p_N - \frac{1}{2}p_S = 0$$

One more equation: $p_R + p_N + p_S = 1$

Solve 3 eqns, 3 unknowns.

$p_R = 0.4$, $p_N = 0.2$, and $p_S = 0.4$

(ii) The steady-state probabilites can also be found in MATLAB via eigen-decomposition.

First, we need to put the matrix P into MATLAB:

```
P = sym([1/2 1/4 1/4; 1/2 0 1/2; 1/4 1/4 1/2]);
```

Method 1:

In class, we have seen how to find the steady-state probabilities via finding the limiting distribution of

$\underline{p}^{(n)} = \underline{p}^{(0)} P^{\cdot\cdot}$ (Alternatively, we may write $\underline{p}^{(n)} = \underline{p}^{(1)} P^{\cdots}$

To find the limit of P^n, we use eigen-decomposition:

```
% Method 1
[V,D] = eig(P)
Dinf = (abs(D)==1)
Pinf = V*Dinf*(V^(-1))
p = Pinf(1,:)
eval(p)
```

```
V =
[ 1, -1,  1]
[ 1,  0, -4]
[ 1,  1,  1]
D =
[ 1,   0,    0]
[ 0, 1/4,    0]
[ 0,   0, -1/4]
Dinf =
      1      0      0
      0      0      0
      0      0      0
Pinf =

     [ 2/5, 1/5, 2/5]
     [ 2/5, 1/5, 2/5]
     [ 2/5, 1/5, 2/5]

p =
[ 2/5, 1/5, 2/5]
ans =
     0.4000     0.2000     0.4000
```

$P = V D V^{-1}$

$P^n = V D^n V^{-1}$

$\lim_{n\to\infty} P^n = V \underbrace{\left(\lim_{n\to\infty} D^n\right)}_{D^\infty} V^{-1} \equiv P^\infty$ =

$D^\infty \equiv \lim_{n\to\infty} D^n$

so, we have

$$\lim_{n\to\infty} \underline{p}^{(n)} = \underline{p}^{(0)} \begin{bmatrix} 2/5 & 1/5 & 2/5 \\ 2/5 & 1/5 & 2/5 \\ 2/5 & 1/5 & 2/5 \end{bmatrix}$$

$$= \left[p_R^{(0)} \quad p_N^{(0)} \quad p_S^{(0)}\right] \begin{bmatrix} 2/5 & 1/5 & 2/5 \\ 2/5 & 1/5 & 2/5 \\ 2/5 & 1/5 & 2/5 \end{bmatrix} = \left[2/5 \quad 1/5 \quad 2/5\right]$$

Method 2:

When the system has already reaches its limiting distribution $\underline{p}$, we expect that, in the next slot/step the system will still having the same distribution. Now, going from one slot/step to the next is simply a multiplication by the transition probability matrix P. Therefore, we must have

$$\underline{p} = \underline{p} P.$$

Equivalently, $$\underline{p}^T = P^T \underline{p}^T.$$

Let's compare this with $\lambda \vec{v} = A \vec{v}$. We see that $\underline{p}^T$ is simply the eigen vector of $A = P^T$ whose corresponding eigen value is $\lambda = 1$:

Note the use of P^T

```
% Method 2
[V D] = eig(P');
k = find(diag(D)==1);
v = V(:,k);
p = (v./sum(v))'
eval(p)
```

← find $\lambda = 1$

← find the corresponding eigen-vector

← Normalize the eigenvector to have $\sum p_i = 1$

```
p =
[ 2/5, 1/5, 2/5]
ans =
     0.4000     0.2000     0.4000
```

(d)

```
n = 1e4;           % The number of slots to be considered
S = [1,2,3];        % Two possible states
P = [1/2 1/4 1/4; 1/2 0 1/2; 1/4 1/4 1/2]; % Transition probability matrix
X1 = 2;            % Initial state

X = MarkovChainGS(n,S,P,X1);

% Approximate the transition probabilities from the simulation
P_sim = []; x = X(1:(n-1)); y = X(2:n);
for k = 1:length(S)
   I = find(x==S(k)); LI = length(I);
   yc = y(I); cond_rel_freq = hist(yc,S)/LI;
   P_sim = [P_sim; cond_rel_freq];
end
P_sim

% Approximate the proportions of time that the states occur
p_sim = hist(X,S)./n
```

```
>> MarkovChain_2014_HW5_Q1_d
P_sim =
   0.4995   0.2553   0.2452
   0.5177        0   0.4823
   0.2583   0.2468   0.4949
p_sim =
   0.4090   0.2008   0.3902
```

Simulation results are close enough to the theoretical results obtained in part (c).

(e) Let today be slot 0. Then $\underline{p}^{(0)} = \begin{array}{c} \begin{array}{ccc} R & N & S \end{array} \\ \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \end{array}$.

(i) $\underline{p}^{(1)} = \underline{p}^{(0)} P = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & 1/4 & 1/4 \\ 1/2 & 0 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{bmatrix} = \begin{array}{c} \begin{array}{ccc} R & N & S \end{array} \\ \begin{bmatrix} 1/4 & \underbrace{1/4} & 1/2 \end{bmatrix} \end{array}$

$\hookrightarrow p_N^{(1)} = 1/4$.

(ii) $\underline{p}^{(2)} = \underline{p}^{(1)} P = \begin{bmatrix} 1/4 & 1/4 & 1/2 \end{bmatrix} \begin{bmatrix} 1/2 & 1/4 & 1/4 \\ 1/2 & 0 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{bmatrix} = \begin{bmatrix} \frac{3}{8} & \underbrace{\frac{3}{16}} & \frac{7}{16} \end{bmatrix}$

$\hookrightarrow p_N^{(2)} = 3/16$.

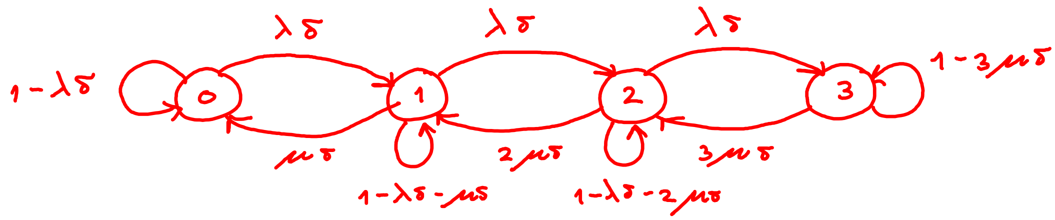
(iii) "365 days" is a long time. $\underline{p}^{(365)} \approx \underline{p} = \begin{bmatrix} 0.4 & 0.2 & 0.4 \end{bmatrix}$

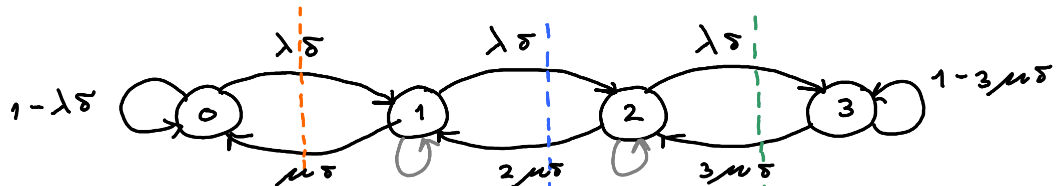
The probability of being a nice day $\approx p_N = 0.2$.

$m = 3$

(a) Erlang B model

Markov chain:





$p_0 \lambda \cancel{\delta} = p_1 \mu \cancel{\delta}$ $\qquad$ $p_1 \cancel{\lambda\delta} = p_2 2\mu \cancel{\delta}$ $\qquad$ $p_2 \lambda \cancel{\delta} = 3\mu \cancel{\delta} p_3$

$p_1 = A p_0$ $\qquad$ $p_2 = \frac{A}{2} p_1 = \frac{A^2}{2} p_0$ $\qquad$ $p_3 = \frac{A}{3} p_2 = \frac{A^3}{3!} p_0$

$$p_0 + p_1 + p_2 + p_3 = 1 \quad \Rightarrow \quad p_0 = \left(1 + A + \frac{A^2}{2} + \frac{A^3}{3}\right)^{-1}$$

$$A = \frac{\lambda}{\mu} = \lambda \times \frac{1}{\mu} = \left(\overset{2}{\cancel{10}} \frac{\text{calls}}{\text{hour}} \times \frac{1\text{ hour}}{\underset{5}{\cancel{60}} \text{ mins}}\right) \times \left(\cancel{12} \text{ mins}\right) = 2 \text{ Erlangs.}$$

$$\Rightarrow p_0 = \frac{3}{19},\ p_1 = \frac{6}{19},\ p_2 = \frac{6}{19},\ \underbrace{p_3 = \frac{4}{19}}$$

$$\Downarrow$$

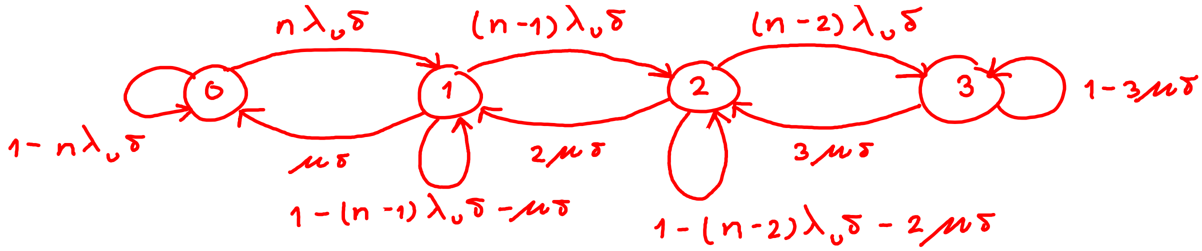
$$\text{call blocking probability} = \frac{4}{19} \approx 0.211$$

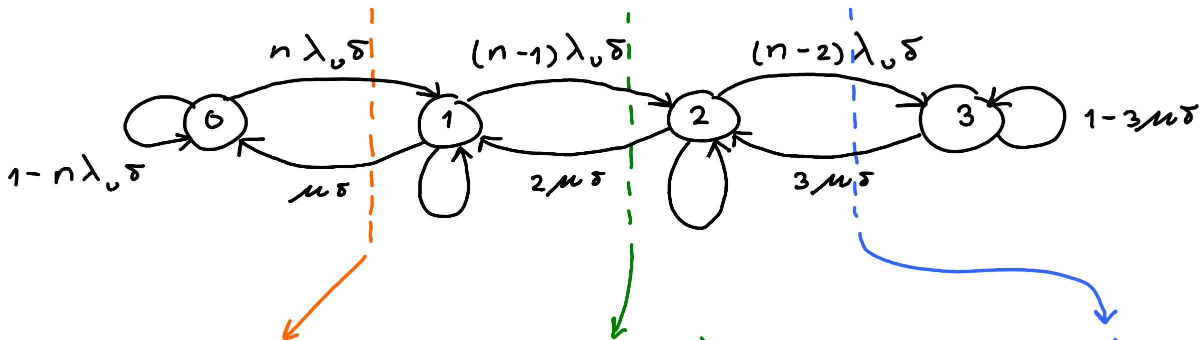
(b) and (c)

Observe that $\lambda_u \times n = \lambda$ in part (a).

So, $\quad \lambda_u = \frac{\lambda}{n}$ and $A_u = \frac{A}{n} = \frac{2}{n}$ Erlangs.

Markov chain:





$p_0 n\lambda_u\cancel{\delta} = p_1\mu\cancel{\delta}$ $\quad$ $p_1(n-1)\lambda_u\cancel{\delta} = 2\mu\cancel{\delta}\,p_2$ $\quad$ $p_2(n-2)\lambda_u\cancel{\delta} = p_3\,3\mu\cancel{\delta}$

$p_1 = nA_u$ $\quad$ $p_2 = \frac{(n-1)}{2}A_u p_1$ $\quad$ $p_3 = \frac{(n-2)}{3}A_u p_2$

$$= \frac{n(n-1)}{2}A_u^2 p_0 \qquad = \frac{n(n-1)(n-2)}{3!}A_u^3 p_0$$

$$p_0+p_1+p_2+p_3 = 1 \Rightarrow \begin{cases} p_0 = \frac{25}{131},\ p_1 = \frac{50}{131},\ p_2 = \frac{40}{131},\ p_3 = \frac{16}{131} & \text{when } n=5, \\ p_0 = 0.159,\ p_1 = 0.319,\ p_2 = 0.316,\ p_3 = 0.206 & \text{when } n=100. \end{cases}$$

As discussed in class, the call blocking probability is given by

$$\frac{(n-m)p_m}{\sum_{k=0}^{m}(n-k)p_k} = \frac{(n-3)p_3}{np_0+(n-1)p_1+(n-2)p_2+(n-3)p_3}$$

$$= \begin{cases} \frac{32}{477} \approx 0.067 & \text{when } n=5 \\ 0.203 & \text{when } n=100 \end{cases}$$

close to the answer from Erlang B.

Remark:

For those who are interested in why the Engset model converges to the Erlang B model when $n\to\infty$, read on.

Note that

$$p_k = \binom{n}{k} A_u^k p_0 = \binom{n}{k} \frac{A^k}{n^k} p_0 = \frac{n!}{(n-k)!\,k!\,n^k} A^k p_0 .$$

For fixed k,

$$\frac{n!}{(n-k)!\,n^k} = \frac{n \times (n-1) \times \cdots \times (n-(k-1))}{n^k} = \frac{n}{n} \times \frac{n-1}{n} \times \cdots \times \frac{n-(k-1)}{n}$$

$$\to 1 \quad \text{as} \quad n \to \infty .$$

Hence,

$$p_k = \frac{\binom{n}{k} \frac{A^k}{n^k}}{\sum_{i=0}^{m} \binom{n}{i} \frac{A^i}{n^i}} \to \underbrace{\frac{\frac{1}{k!} A^k}{\sum_{i=0}^{m} \frac{1}{i!} A^i}} \quad \text{as} \quad n \to \infty$$

same as the steady-state probabilities in Erlang B model.

Similarly, for the call blocking probability,

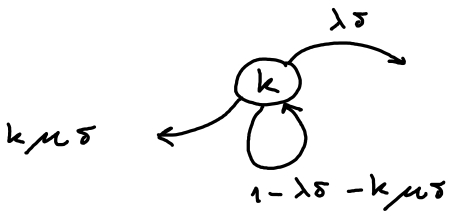
$$P_{CB} = \frac{(n-m)\, p_m}{\sum_{k=0}^{m} (n-k)\, p_k} = \frac{(n-m) \binom{n}{m} \frac{A^m}{n^m} \cancel{p_0}}{\sum_{k=0}^{m} (n-k) \binom{n}{k} \frac{A^k}{n^k} \cancel{p_0}} = \frac{\binom{n}{m} \frac{A^m}{n^m}}{\sum_{k=0}^{m} \frac{n-k}{n-m} \binom{n}{k} \frac{A^k}{n^k}}$$

$$\to \underbrace{\frac{\frac{A^m}{m!}}{\sum_{k=0}^{m} \frac{A^k}{k!}}} \quad \text{as} \quad n \to \infty$$

same as the call blocking probability in Erlang B model.

(a) Nothing changes from the M/M/m/m model when $k < m$.

We have

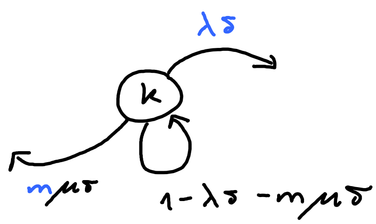


When $k \geq m$, the call request rate is still λ. The difference is that now we have a queue for the new requests to wait. (In M/M/m/m, these requests are discarded and the calls are blocked.)

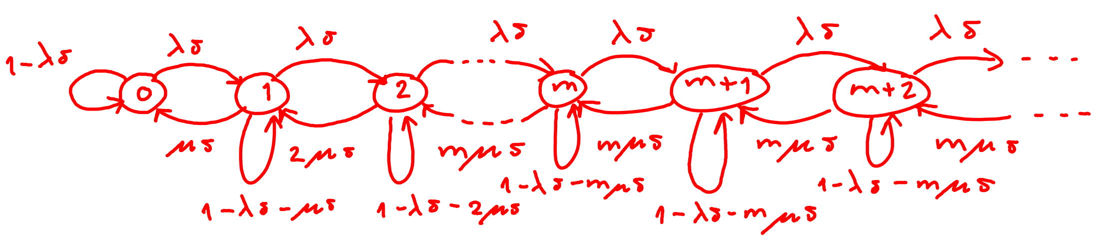
When $k \geq m$, all m channels are being used. There are $k-m$ requests waiting in the queue. When there is one new call request, it will be added to the queue and hence the system move from state k to $k+1$. Again, this new call request occurs with probability $\lambda\delta$ (approximately).

When $k \geq m$, all m channels are being used. There are m customers talking on the phone. So, the probability of one call ends is (approximately) $m\mu\delta$.

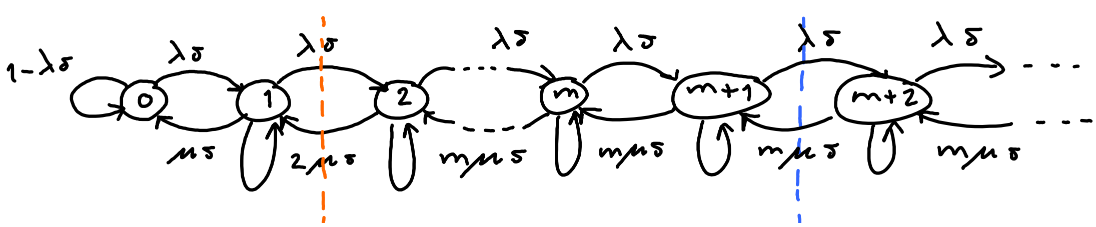
Therefore, when $k \geq m$, we have

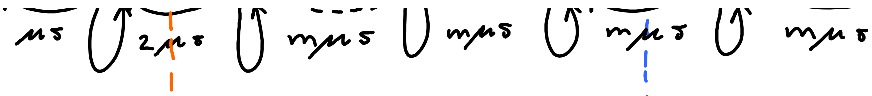


Markov chain:



(b)





$$p_{k-1}\lambda\cancel{\delta} = p_k k\mu\cancel{\delta}$$

$$p_k = \frac{A}{k}p_{k-1}$$

$$= \frac{A^k}{k!}p_0$$

for $0<k<m$

$$p_{k-1}\lambda\cancel{\delta} = m\mu\cancel{\delta}\, p_k$$

$$p_k = \frac{A}{m}p_{k-1}$$

for $k \geqslant m$

$$\Rightarrow p_m = \frac{A}{m}p_{m-1}$$

$$= \frac{A}{m}\frac{A^{m-1}}{(m-1)!}p_0$$

$$= \frac{A^m}{m!}p_0$$

$$p_k = \left(\frac{A}{m}\right)^{k-m}\frac{A^m}{m!}p_0 = \frac{A^k}{m!(m^{k-m})}p_0$$

$$= \frac{m^m}{m!}\left(\frac{A}{m}\right)^k p_0 \quad \text{for } k \geqslant m.$$

$$\sum_{k=0}^{m} p_k = 1 \Rightarrow 1 = \sum_{k=0}^{m-1}\frac{A^k}{k!}p_0 + \underbrace{\sum_{k=m}^{\infty}\frac{m^m}{m!}\left(\frac{A}{m}\right)^k p_0} = \sum_{k=0}^{m-1}\frac{A^k}{k!}p_0 + \frac{A^m}{m!\left(1-\frac{A}{m}\right)}p_0$$

geometric series

$$= \begin{cases} \frac{m^m}{m!}\frac{\left(\frac{A}{m}\right)^m}{1-\frac{A}{m}}p_0 & \text{if } A<m \\ \infty & \text{if } A \geqslant m \end{cases}$$

$$\Rightarrow p_0 = \left(\left(\sum_{k=0}^{m-1}\frac{A^k}{k!}\right) + \frac{A^m}{m!\left(1-\frac{A}{m}\right)}\right)^{-1}$$

Therefore,

$$p_k = \begin{cases} \frac{A^k}{k!}p_0, & k<m \\ \frac{A^k}{m!(m^{k-m})}p_0, & k \geqslant m \end{cases}$$

(c) Delayed call probability

$$= \sum^{\infty} p_k = \frac{A^m}{\ldots} \quad p_0 = \frac{\frac{A^m}{m!(1-A/m)}}{\ldots}$$

$$= \sum_{k=m}^{\infty} p_k = \frac{A^m}{m!\left(1-\frac{A}{m}\right)} p_0 = \frac{\overline{m!\,(1-A/m)}}{\frac{A^m}{m!\,(1-\frac{A}{m})} + \sum_{k=0}^{m-1} \frac{A^k}{k!}}$$

$$= \frac{A^m}{A^m + m!\left(1-\frac{A}{m}\right)\sum_{k=0}^{m-1} \frac{A^k}{k!}}$$

Remark: This formula is call the "Erlang C formula".

ECS455 HW5 Q4 More on Engset Model (Optional)

Wednesday, January 05, 2011 3:21 PM

(a)

$$P_m = \frac{\binom{n}{m} A_u^m}{\sum_{k=0}^{m} \binom{n}{k} A_u^k} = \frac{\sum_{k=0}^{m} \binom{n}{k} A_u^k - \sum_{k=0}^{m-1} \binom{n}{k} A_u^k}{\sum_{k=0}^{m} \binom{n}{k} A_u^k}$$

$$= \frac{z(m,n) - z(m-1,n)}{z(m,n)} = 1 - \frac{z(m-1,n)}{z(m,n)}$$

Hence, $c = 1$

(b)

First, note that

$$(n-k) \times \binom{n}{k} = (n-k) \times \frac{n!}{k!(n-k)!} = \frac{n!}{k!(n-k-1)!}$$

$$= \frac{n \times (n-1)!}{k!\,(n-1-k)!} = n \binom{n-1}{k}$$

Therefore,

$$P_b = \frac{(n-m)\binom{n}{m} A_u^m}{\sum_{k=0}^{m} (n-k)\binom{n}{k} A_u^k} = \frac{\cancel{n}\binom{n-1}{m} A_u^m}{\sum_{k=0}^{m} \cancel{n}\binom{n-1}{k} A_u^k}$$

$$= \frac{\sum_{k=0}^{m} \binom{n-1}{k} A_u^k - \sum_{k=0}^{m-1} \binom{n-1}{k} A_u^k}{\sum_{k=0}^{m} \binom{n-1}{k} A_u^k}$$

$$= \frac{z(m,n-1) - z(m-1,n-1)}{z(m,n-1)} = 1 - \frac{z(m-1,n-1)}{z(m,n-1)}$$

Hence, $c_1 = c_2 = c_4 = 1$ and $c_3 = 0$.

(c)

(c)

$$P_b = \frac{\binom{n-1}{m} A_u^m}{\sum_{k=0}^{m} \binom{n-1}{k} A_u^k} \overset{m=n-1}{=} \frac{\binom{m}{m} A_u^m}{\sum_{k=0}^{m} \binom{m}{k} A_u^k} = \frac{A_u^m}{(1+A_u)^m} = \left(\frac{A_u}{1+A_u}\right)^m$$